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SOME COEFFICIENT INEQUALITIES AND NEIGHBORHOOD PROPERTIES ASSOCIATED WITH ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Abstract

The *main* purpose of this lecture is to present some interesting recent developments concerning coefficient and distortion inequalities, neighborhood properties, and majorization problems associated with certain families of analytic and multivalent functions. Some of the various analytic function classes, which are considered in this lecture, are defined by means of the familiar Ruscheweyh derivative and a certain *nonhomogeneous* Cauchy-Euler differential equation. Several analytic function classes of *complex* order are also investigated.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{T}(n,p)$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \qquad (a_k \geqq 0;\ p,n \in \mathbb{N} := \{1,2,3,\ldots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Following the earlier investigations by Goodman [13] and Ruscheweyh [25] (see also Silverman [27] and Altintaş *et al.* ([6], [7], and [9])), we define the (n,δ)-neighborhood of a function $f(z) \in \mathcal{T}(n,p)$ by

$$N_{n,\delta}(f;g) := \left\{ g \in \mathcal{T}(n,p) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leqq \delta \right\}, \tag{1.2}$$

so that, obviously,

$$N_{n,\delta}(h;g) := \left\{ g \in \mathcal{T}(n,p) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k\,|b_k| \leqq \delta \right\}, \tag{1.3}$$

where

$$h(z) = z^p \qquad (p \in \mathbb{N}). \tag{1.4}$$

First of all, we denote by $\mathcal{S}_n^*(p,\alpha)$ and $\mathcal{C}_n(p,\alpha)$ the classes of *p-valently starlike functions of order* α in $\mathbb{U}$ $(0 \leqq \alpha < p)$ and *p-valently convex functions of order* α in $\mathbb{U}$ $(0 \leqq \alpha < p)$, respectively. Thus, by definition, we have

$$\mathcal{S}_n^*(p,\alpha) := \left\{ f \in \mathcal{T}(n,p) : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < p) \right\} \tag{1.5}$$

and

$$\mathcal{C}_n(p,\alpha) := \left\{ f \in \mathcal{T}(n,p) : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < p) \right\}. \tag{1.6}$$

An interesting unification of the function classes $\mathcal{S}_n^*(p,\alpha)$ and $\mathcal{C}_n(p,\alpha)$ is provided by the class $\mathcal{T}_n(p,\alpha,\lambda)$ of functions $f \in \mathcal{T}(n,p)$, which also satisfy the following inequality:

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right) > \alpha \tag{1.7}$$

$$(z \in \mathbb{U};\ 0 \leqq \alpha < p;\ 0 \leqq \lambda \leqq 1).$$

The class $\mathcal{T}_n(p,\alpha,\lambda)$ was investigated by Altintaş *et al.* [4] and (subsequently) by Irmak *et al.* [15]. In particular, the class $\mathcal{T}_n(1,\alpha,\lambda)$ was considered earlier by Altintaş [3]. Clearly, we have

$$\mathcal{T}_n(p,\alpha,0) = \mathcal{S}_n^*(p,\alpha) \quad \text{and} \quad \mathcal{T}_n(p,\alpha,1) = \mathcal{C}_n(p,\alpha) \tag{1.8}$$

in terms of the *simpler* classes $\mathcal{S}_n^*(p,\alpha)$ and $\mathcal{C}_n(p,\alpha)$ defined by (1.5) and (1.6), respectively (see also Duren [12], Goodman [14], and Srivastava and Owa ([28] and [29])).

Based substantially upon a sequel to the aforementioned recent works by Altintaş *et al.* [9], we begin our investigation here by presenting several coefficient inequalities and distortion bounds, and associated inclusion relations for the (n,δ)-neighborhood of functions in the subclass $\mathcal{K}_n(p,\alpha,\lambda,\mu)$ of the class $\mathcal{T}(n,p)$, which consists of functions $f \in \mathcal{T}(n,p)$ satisfying the following *nonhomogeneous* Cauchy-Euler differential equation:

$$z^2\frac{d^2w}{dz^2} + 2(\mu+1)\,z\frac{dw}{dz} + \mu(\mu+1)\,w = (p+\mu)(p+\mu+1)\,g(z) \tag{1.9}$$

$$\left(w = f(z) \in \mathcal{T}(n,p);\ g \in \mathcal{T}_n(p,\alpha,\lambda);\ \mu > -p\ \ (\mu \in \mathbb{R})\right).$$

We shall also investigate, in our presentation here, several other univalent and multivalent analytic function classes [defined by means of (for example) the familiar Ruscheweyh derivative] as well as the majorization problems associated with some of these analytic function classes.

2. Coefficient Inequalities, Distortion Bounds, and Neighborhood Properties for the Classes $\mathcal{T}_n(p,\alpha,\lambda)$ and $\mathcal{K}_n(p,\alpha,\lambda,\mu)$

Lemma 1 and Lemma 2 below are remarkably instrumental in establishing the *main* distortion bounds for functions in the class $\mathcal{K}_n(p,\alpha,\lambda,\mu)$, given by Theorem 1.

Lemma 1 (Altintaş *et al.* [4, p. 10, Theorem 1]). *Let the function $f \in \mathcal{T}(n,p)$ be defined by* (1.1). *Then the function $f(z)$ is in the class $\mathcal{T}_n(p,\alpha,\lambda)$ if and only if*

$$\sum_{k=n+p}^{\infty} (k-\alpha)\left[\lambda(k-1)+1\right] a_k \leqq (p-\alpha)\left[\lambda(p-1)+1\right] \tag{2.1}$$

$$\left(0 \leqq \alpha < p;\ 0 \leqq \lambda \leqq 1;\ n,p \in \mathbb{N}\right).$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{(p-\alpha)\left[\lambda(p-1)+1\right]}{(n+p-\alpha)\left[\lambda(n+p-1)+1\right]}\, z^{n+p} \qquad (n,p \in \mathbb{N}). \tag{2.2}$$

Lemma 2. (Altintaş *et al.* [9]). *Let the function $f(z)$ given by* (1.1) *be in the class $\mathcal{T}_n(p,\alpha,\lambda)$. Then*

$$\sum_{k=n+p}^{\infty} a_k \leqq \frac{(p-\alpha)\left[\lambda(p-1)+1\right]}{(n+p-\alpha)\left[\lambda(n+p-1)+1\right]} \tag{2.3}$$

and

$$\sum_{k=n+p}^{\infty} ka_k \leqq \frac{(n+p)(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]}. \tag{2.4}$$

Theorem 1. *If $f \in \mathcal{T}(n,p)$ is in the class $\mathcal{K}_n(p,\alpha,\lambda,\mu)$, then*

$$|f(z)| \leqq |z|^p + \frac{(p-\alpha)[\lambda(p-1)+1](p+\mu)(p+\mu+1)}{(n+p-\alpha)[\lambda(n+p-1)+1](n+p+\mu)}|z|^{n+p} \qquad (z \in \mathbb{U}) \tag{2.5}$$

and

$$|f(z)| \geqq |z|^p - \frac{(p-\alpha)[\lambda(p-1)+1](p+\mu)(p+\mu+1)}{(n+p-\alpha)[\lambda(n+p-1)+1](n+p+\mu)}|z|^{n+p} \qquad (z \in \mathbb{U}). \tag{2.6}$$

Proof. Suppose that $f \in \mathcal{T}(n,p)$ is given by (1.1). Also let the function $g \in \mathcal{T}_n(p,\alpha,\lambda)$, occurring in the *nonhomogeneous* Cauchy-Euler differential equation (1.9), be given as in the definitions (1.2) and (1.3) *with*, of course,

$$b_k \geqq 0 \qquad (k = n+p, n+p+1, n+p+2, \ldots). \tag{2.7}$$

Then we readily find from (1.9) that

$$a_k = \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}\, b_k \qquad (k = n+p, n+p+1, n+p+2, \ldots), \tag{2.8}$$

so that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k\, z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}\, b_k\, z^k \tag{2.9}$$

and

$$|f(z)| \leqq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}\, b_k \qquad (z \in \mathbb{U}). \tag{2.10}$$

Next, since $g \in \mathcal{T}_n(p,\alpha,\lambda)$, the first assertion (2.3) of Lemma 2 yields the coefficient inequality:

$$b_k \leqq \frac{(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} \qquad (k = n+p, n+p+1, n+p+3, \ldots), \tag{2.11}$$

which, in conjunction with (2.10), yields

$$|f(z)| \leqq |z|^p + \frac{(p-\alpha)[\lambda(p-1)+1](p+\mu)(p+\mu+1)}{(n+p-\alpha)[\lambda(n+p-1)+1]}|z|^{n+p} \cdot \sum_{k=n+p}^{\infty} \frac{1}{(k+\mu)(k+\mu+1)} \qquad (z \in \mathbb{U}). \tag{2.12}$$

Finally, in view of the *telescopic* sum:

$$\sum_{k=n+p}^{\infty} \frac{1}{(k+\mu)(k+\mu+1)} = \sum_{k=n+p}^{\infty} \left(\frac{1}{k+\mu} - \frac{1}{k+\mu+1} \right) = \frac{1}{n+p+\mu} \tag{2.13}$$

$$(\mu \in \mathbb{R} \setminus \{-n-p, -n-p-1, -n-p-2, \ldots\}),$$

the first assertion (2.5) of Theorem 1 follows at once from (2.12).

The second assertion (2.6) of Theorem 1 can be proven by similarly applying (2.9), (2.11), and (2.13).

By setting $\lambda = 0$ and $\lambda = 1$ in Theorem 1, and using the relationships in (1.8), we arrive at Corollary 1 and Corollary 2, respectively.

Corollary 1. *If the functions f and g satisfy the nonhomogeneous Cauchy-Euler differential equation* (1.9) *with $g \in \mathcal{S}_n^* (p, \alpha)$, then*

$$\begin{aligned} |z|^p - \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)} |z|^{n+p} &\leqq |f(z)| \\ &\leqq |z|^p + \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)} |z|^{n+p} \qquad (z \in \mathbb{U}). \end{aligned} \tag{2.14}$$

Corollary 2. *If the functions f and g satisfy the nonhomogeneous Cauchy-Euler differential equation* (1.9) *with $g \in \mathcal{C}_n (p, \alpha)$, then*

$$\begin{aligned} |z|^p - \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)(n+p-\alpha)(n+p+\mu)} |z|^{n+p} &\leqq |f(z)| \\ &\leqq |z|^p + \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)(n+p-\alpha)(n+p+\mu)} |z|^{n+p} \qquad (z \in \mathbb{U}). \end{aligned} \tag{2.15}$$

Now we turn to the determination of the inclusion relations for the classes $\mathcal{T}_n(p, \alpha, \lambda)$ and $\mathcal{K}_n(p, \alpha, \lambda, \mu)$ involving the (n, δ)-neighborhoods defined by (1.2) and (1.3). We first state

Theorem 2. *If $f \in \mathcal{T}(n, p)$ is in the class $\mathcal{T}_n(p, \alpha, \lambda)$, then*

$$\mathcal{T}_n(p, \alpha, \lambda) \subset N_{n,\delta}(h; f), \tag{2.16}$$

where $h(z)$ is given by (1.4) *and*

$$\delta := \frac{(n+p)(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]}. \tag{2.17}$$

Proof. The assertion (2.16) would follow easily from the definition of $N_{n,\delta}(h; f)$, which is given by (1.3) with $g(z)$ replaced by $f(z)$, and the second assertion (2.4) of Lemma 2.

Theorem 3. *If $f \in \mathcal{T}(n,p)$ is in the class $\mathcal{K}_n(p,\alpha,\lambda,\mu)$, then*

$$\mathcal{K}_n(p,\alpha,\lambda,\mu) \subset N_{n,\delta}(g;f), \tag{2.18}$$

where $g(z)$ is given by (1.9) *and*

$$\delta := \frac{(n+p)(p-\alpha)[\lambda(p-1)+1][n+(p+\mu)(p+\mu+2)]}{(n+p-\alpha)[\lambda(n+p-1)+1](n+p+\mu)}. \tag{2.19}$$

Proof. Suppose that $f \in \mathcal{K}_n(p,\alpha,\lambda,\mu)$. Then, upon substituting from (2.8) into the coefficient inequality:

$$\sum_{k=n+p}^{\infty} k\,|b_k - a_k| \leqq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} ka_k \qquad (a_k \geqq 0;\ b_k \geqq 0), \tag{2.20}$$

we obtain

$$\sum_{k=n+p}^{\infty} k\,|b_k - a_k| \leqq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}\, kb_k \tag{2.21}$$

Next, since $g \in \mathcal{T}_n(p,\alpha,\lambda)$, the second assertion (2.4) of Lemma 2 yields

$$kb_k \leqq \frac{(n+p)(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} \qquad (k = n+p, n+p+1, n+p+2, \ldots). \tag{2.22}$$

Finally, by making use of (2.4) as well as (2.22) on the right-hand side of (2.21), we find that

$$\sum_{k=n+p}^{\infty} k\,|b_k - a_k| \leqq \frac{(n+p)(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} \left(1 + \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}\right), \tag{2.23}$$

which, by virtue of the *telescopic* sum (2.13), immediately yields

$$\sum_{k=n+p}^{\infty} k\,|b_k - a_k| \leqq \frac{(n+p)(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} \left(\frac{n+(p+\mu)(p+\mu+2)}{n+p+\mu}\right) =: \delta. \tag{2.24}$$

Thus, by the definition (1.2) with $g(z)$ interchanged by $f(z)$, $f \in N_{n,\delta}(g;f)$. This evidently completes the proof of Theorem 2.

3. Further Neighborhood Properties Involving Analytic Functions with Negative and Missing Coefficients

We denote by $\mathcal{T}(n) := \mathcal{T}(n,1)$ the class of functions f of the form [*cf.* Equation (1.1)]:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geqq 0;\ n \in \mathbb{N}), \tag{3.1}$$

which are *analytic* in the *open* unit disk $\mathbb{U}$. And, just as in Definitions (1.2) and (1.3), we define the (n,δ)-neighborhood of a function $f \in \mathcal{T}(n)$ by

$$N_{n,\delta}(f) := \left\{ g \in \mathcal{T}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k \, z^k \text{ and } \sum_{k=n+1}^{\infty} k \, |a_k - b_k| \leqq \delta \right\}. \tag{3.2}$$

In particular, for the *identity* function

$$e(z) = z, \tag{3.3}$$

we immediately have

$$N_{n,\delta}(e) := \left\{ g \in \mathcal{T}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k \, z^k \text{ and } \sum_{k=n+1}^{\infty} k \, |b_k| \leqq \delta \right\}. \tag{3.4}$$

The above concept of (n,δ)-neighborhoods was extented and applied recently to families of *analytically* multivalent functions by Altintaş *et al.* [9] and to families of *meromorphically* multivalent functions by Liu and Srivastava ([16] and [17]). In this section, we investigate the (n,δ)-neighborhoods of several subclasses of the class $\mathcal{T}(n)$ of *normalized* analytic functions in $\mathbb{U}$ with negative *and* missing coefficients, which are introduced below by making use of the familiar Ruscheweyh derivative (see, for details, Murugusundaramoorthy and Srivastava [20]; see also Ahuja and Nunokawa [2], Ruscheweyh [24], and others).

First of all, we say that a function $f \in \mathcal{T}(n)$ is *starlike of complex order* γ $(\gamma \in \mathbb{C} \setminus \{0\})$, that is, $f \in \mathcal{S}_n^*(\gamma)$, if it also satisfies the following inequality:

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0 \quad (z \in \mathbb{U};\ \gamma \in \mathbb{C} \setminus \{0\}). \tag{3.5}$$

Furthermore, a function $f \in \mathcal{T}(n)$ is said to be *convex of complex order* γ $(\gamma \in \mathbb{C} \setminus \{0\})$, that is, $f \in \mathcal{C}_n(\gamma)$, if it also satisfies the following inequality:

$$\Re\left(1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U};\ \gamma \in \mathbb{C} \setminus \{0\}). \tag{3.6}$$

The classes $\mathcal{S}_n^*(\gamma)$ and $\mathcal{C}_n(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf [21] and Wiatrowski [30], respectively (see also Altintaş *et al.* ([8] and [10])).

Next, for the functions f_j $(j = 1,2)$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} \, z^k \qquad (j = 1,2), \tag{3.7}$$

let $f_1 * f_2$ denote the Hadamard product (or convolution) of f_1 and f_2, defined by

$$(f_1 * f_2)(z) := z + \sum_{k=2}^{\infty} a_{k,1} \, a_{k,2} \, z^k =: (f_2 * f_1)(z). \tag{3.8}$$

Thus the Ruscheweyh derivative operator $D^\lambda : \mathcal{T} \longrightarrow \mathcal{T}$ is defined for $\mathcal{T} := \mathcal{T}(1)$ by

$$D^\lambda f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda > -1;\ f \in \mathcal{T}) \tag{3.9}$$

or, equivalently, by

$$D^\lambda\, f(z) := z - \sum_{k=2}^{\infty} \binom{\lambda+k-1}{k-1} a_k\, z^k \qquad (\lambda > -1;\ f \in \mathcal{T}) \tag{3.10}$$

for a function $f \in \mathcal{T}$ of the form (3.1). Here, *and in what follows*, we make use of the following standard notation:

$$\binom{\kappa}{k} := \frac{\kappa(\kappa-1)\cdots(\kappa-k+1)}{k!} \qquad (\kappa \in \mathbb{C};\ k \in \mathbb{N}_0) \tag{3.11}$$

for a binomial coefficient. In particular, we have

$$D^n\, f(z) = \frac{z\left(z^{n-1}\, f(z)\right)^{(n)}}{n!} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{3.12}$$

Finally, in terms of the Ruscheweyh derivative operator $D^\lambda\,(\lambda > -1)$ defined by (3.9) or (3.10) above, let $\mathcal{S}_n(\gamma,\lambda,\beta)$ denote the subclass of $\mathcal{T}(n)$ consisting of functions f which satisfy the following inequality:

$$\left| \frac{1}{\gamma}\left(\frac{z\,(D^\lambda\, f(z))'}{D^\lambda\, f(z)} - 1 \right) \right| < \beta \tag{3.13}$$

$$(z \in \mathbb{U};\ \gamma \in \mathbb{C}\setminus\{0\};\ \lambda > -1;\ 0 < \beta \leqq 1).$$

Also let $\mathcal{R}_n(\gamma,\lambda,\beta;\mu)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions f which satisfy the following inequality:

$$\left| \frac{1}{\gamma}\left((1-\mu)\,\frac{D^\lambda\, f(z)}{z} + \mu\,(D^\lambda\, f(z))' - 1 \right) \right| < \beta \tag{3.14}$$

$$(z \in \mathbb{U};\ \gamma \in \mathbb{C}\setminus\{0\};\ \lambda > -1;\ 0 < \beta \leqq 1;\ \mu \geqq 0).$$

Various *further* subclasses of the classes $\mathcal{S}_n(\gamma,\lambda,\beta)$ and $\mathcal{R}_n(\gamma,\lambda,\beta;\mu)$ *with* $\gamma = 1$ were studied in many earlier works (*cf.*, *e.g.*, Duren [12], Goodman [14], and Srivastava and Owa ([28] and [29]); see also the references cited in these earlier works). Clearly, in the case of (for example) the class $\mathcal{S}_n(\gamma,\lambda,\beta)$, we have

$$\mathcal{S}_n(\gamma,0,1) \subset \mathcal{S}_n^*(\gamma) \quad \text{and} \quad \mathcal{S}_n(\gamma,1,1) \subset \mathcal{C}_n(\gamma) \tag{3.15}$$

$$(n \in \mathbb{N};\ \gamma \in \mathbb{C}\setminus\{0\}).$$

In our investigation of the inclusion relations involving $N_{n,\delta}(e)$, we shall require Lemma 3 and Lemma 4 below.

Lemma 3 (Murugusundaramoorthy and Srivastava [20]). *Let the function $f \in \mathcal{A}(n)$ be defined by* (3.1). *Then f is in the class $\mathcal{S}_n(\gamma, \lambda, \beta)$ if and only if*

$$\sum_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} (\beta|\gamma| + k - 1)\, a_k \leqq \beta|\gamma|. \tag{3.16}$$

Proof. We first suppose that $f \in \mathcal{S}_n(\gamma, \lambda, \beta)$. Then, by appealing to the condition (3.13), we readily obtain

$$\Re\left(\frac{z\,(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right) > -\beta|\gamma| \quad (z \in \mathbb{U}) \tag{3.17}$$

or, equivalently,

$$\Re\left(\frac{-\sum\limits_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1}(k-1)a_k z^k}{z - \sum\limits_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} a_k z^k}\right) > -\beta|\gamma| \quad (z \in \mathbb{U}), \tag{3.18}$$

where we have made use of (3.10) *and* the definition (3.1).

We now choose values of z on the real axis and let $z \to 1-$ through real values. Then the inequality (3.18) immediately yields the desired condition (3.16).

Conversely, by applying the hypothesis (3.16) and letting $|z| = 1$, we find that

$$\begin{aligned}\left|\frac{z\,(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right| &= \left|\frac{\sum\limits_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1}(k-1)a_k z^k}{z - \sum\limits_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} a_k z^k}\right| \\ &\leqq \frac{\beta|\gamma| \left(1 - \sum\limits_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} a_k\right)}{1 - \sum\limits_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} a_k} \\ &\leqq \beta|\gamma|. \end{aligned} \tag{3.19}$$

Hence, by the *maximum modulus theorem*, we have

$$f \in \mathcal{S}_n(\gamma, \lambda, \beta),$$

which evidently completes the proof of Lemma 3.

Similarly, we can prove the following result.

Lemma 4 (*cf.* Murugusundaramoorthy and Srivastava [20]). *Let the function $f \in \mathcal{A}(n)$ be defined by* (3.1). *Then f is in the class $\mathcal{R}(\gamma, \lambda, \beta; \mu)$ if and only if*

$$\sum_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} [\mu(k-1)+1]\, a_k \leqq \beta|\gamma|. \tag{3.20}$$

Remark 1. A special case of Lemma 3 when

$$n=1, \qquad \gamma=1, \qquad \text{and} \qquad \beta = 1-\alpha \qquad (0 \leqq \alpha < 1)$$

was given earlier by Ahuja [1]. Furthermore, if in Lemma 3 with

$$n=1, \qquad \gamma=1, \qquad \text{and} \qquad \beta = 1-\alpha \qquad (0 \leqq \alpha < 1),$$

we set $\lambda = 0$ and $\lambda = 1$, we shall obtain the familiar *earlier* results of Silverman [26].

The first inclusion relation involving $N_{n,\delta}(e)$ is given by Theorem 4 below.

Theorem 4. *If*

$$\delta := \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)\binom{\lambda+n}{n}} \qquad (|\gamma|<1), \tag{3.21}$$

then

$$\mathcal{S}_n(\gamma, \lambda, \beta) \subset N_{n,\delta}(e). \tag{3.22}$$

Proof. For a function $f \in \mathcal{S}_n(\gamma, \lambda, \beta)$ of the form (3.1), Lemma 3 immediately yields

$$(\beta|\gamma|+n)\binom{\lambda+n}{n} \sum_{k=n+1}^{\infty} a_k \leqq \beta|\gamma|,$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{\beta|\gamma|}{(\beta|\gamma|+n)\binom{\lambda+n}{n}}. \tag{3.23}$$

On the other hand, we also find from (3.16) and (3.23) that

$$\begin{aligned}
\binom{\lambda+n}{n} \sum_{k=n+1}^{\infty} k a_k &\leqq \beta|\gamma| + (1-\beta|\gamma|)\binom{\lambda+n}{n} \sum_{k=n+1}^{\infty} a_k \\
&\leqq \beta|\gamma| + (1-\beta|\gamma|)\binom{\lambda+n}{n} \frac{\beta|\gamma|}{(\beta|\gamma|+n)\binom{\lambda+n}{n}} \\
&\leqq \frac{(n+1)\beta|\gamma|}{\beta|\gamma|+n} \qquad (|\gamma|<1),
\end{aligned}$$

that is,

$$\sum_{k=n+1}^{\infty} ka_k \leqq \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)\binom{\lambda+n}{n}} := \delta, \tag{3.24}$$

which, in view of the definition (3.4), proves Theorem 4.

By similarly applying Lemma 4 instead of Lemma 3, we now prove Theorem 5 below.

Theorem 5. *If*

$$\delta := \frac{(n+1)\,\beta\,|\gamma|}{(\mu n+1)\binom{\lambda+n}{n}} \qquad (\mu > 1), \tag{3.25}$$

then

$$\mathcal{R}_n(\gamma,\lambda,\beta;\mu) \subset N_{n,\delta}(e). \tag{3.26}$$

Proof. Suppose that a function $f \in \mathcal{R}(\gamma,\lambda,\beta;\mu)$ is of the form (3.1). Then we find from the assertion (3.20) of Lemma 4 that

$$\binom{\lambda+n}{n}(\mu n+1)\sum_{k=n+1}^{\infty} a_k \leqq \beta\,|\gamma|,$$

which yields the following coefficient inequality:

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{\beta\,|\gamma|}{(\mu n+1)\binom{\lambda+n}{n}}. \tag{3.27}$$

Finally, by making use of (3.20) in conjunction with (3.27), we also have

$$\begin{aligned} \mu\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty} ka_k &\leqq \beta\,|\gamma| + (\mu-1)\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty} a_k \\ &\leqq \beta\,|\gamma| + (\mu-1)\binom{\lambda+n}{n}\frac{\beta\,|\gamma|}{(\mu n+1)\binom{\lambda+n}{n}}, \end{aligned}$$

that is,

$$\sum_{k=n+1}^{\infty} ka_k \leqq \frac{(n+1)\,\beta\,|\gamma|}{(\mu n+1)\binom{\lambda+n}{n}} =: \delta,$$

which, in light of the definition (3.4), completes the proof of Theorem 5.

Remark 2. By suitably specializing the various parameters involved in Theorem 4 and Theorem 5, we can derive the corresponding inclusion relations for many relatively more familiar function classes (see also Equation (3.15) *and* Remark 1 above).

Next we determine the neighborhood for each of the function classes

$$\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta) \quad \text{and} \quad \mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta;\mu),$$

which we define as follows. A function $f \in \mathcal{T}(n)$ is said to be in the class $\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta)$ if there exists a function $g \in \mathcal{S}_n(\gamma,\lambda,\beta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{3.28}$$

Analogously, a function $f \in \mathcal{T}(n)$ is said to be in the class $\mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta;\mu)$ if there exists a function $g \in \mathcal{R}_n(\gamma,\lambda,\beta;\mu)$ such that the inequality (3.28) holds true.

Theorem 6. *If* $g \in \mathcal{S}_n(\gamma,\lambda,\beta)$ *and*

$$\alpha = 1 - \frac{(\beta\,|\gamma| + n)\,\delta\binom{\lambda+n}{n}}{(n+1)\left[(\beta\,|\gamma| + n)\binom{\lambda+n}{n} - \beta\,|\gamma|\right]}, \tag{3.29}$$

then

$$N_{n,\delta}(g) \subset \mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta). \tag{3.30}$$

Proof. Suppose that $f \in N_{n,\delta}(g)$. We then find from the definition (3.2) that

$$\sum_{k=n+1}^{\infty} k\,|a_k - b_k| \leqq \delta, \tag{3.31}$$

which readily implies the coefficient inequality:

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leqq \frac{\delta}{n+1} \quad (n \in \mathbb{N}). \tag{3.32}$$

Next, since $g \in \mathcal{S}_n(\gamma,\lambda,\beta)$, we have [*cf.* Equation (3.23)]

$$\sum_{k=n+1}^{\infty} b_k \leqq \frac{\beta|\gamma|}{(\beta\,|\gamma| + n)\binom{\lambda+n}{n}}, \tag{3.33}$$

so that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum\limits_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=n+1}^{\infty}b_k}$$

$$\leqq \frac{\delta}{n+1}\cdot\frac{(\beta|\gamma|+n)\binom{\lambda+n}{n}}{(\beta|\gamma|+n)\binom{\lambda+n}{n}-\beta|\gamma|}$$

$$=1-\alpha, \tag{3.34}$$

provided that α is given precisely by (3.29). Thus, by definition, $f\in\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta)$ for α given by (3.29). This evidently completes our proof of Theorem 6.

The proof of Theorem 7 below is much akin to that of Theorem 6.

Theorem 7. *If* $g\in\mathcal{R}_n(\gamma,\lambda,\beta;\mu)$ *and*

$$\alpha=1-\frac{(\mu n+1)\,\delta\binom{\lambda+n}{n}}{(n+1)\left[(\mu n+1)\binom{\lambda+n}{n}-\beta|\gamma|\right]}, \tag{3.35}$$

then

$$N_{n,\delta}(g)\subset\mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta;\mu). \tag{3.36}$$

Remark 3. Just as we already indicated in (especially) Remark 2, Theorem 6 and Theorem 7 can readily be specialized to deduce the corresponding neighborhood properties for many simpler function classes.

4. Majorization Problems Associated with p-Valently Starlike and Convex Functions of Complex Order

In this last section of our presentation here, we propose to investigate several majorization problems involving two interesting subclasses of *p-valently starlike* and *p-valently convex functions of complex order* $\gamma\neq 0$ in the open unit disk $\mathbb{U}$.

Suppose that the functions $f(z)$ and $g(z)$ are *analytic* in the *open* unit disk

$$\mathbb{U}:=\{z: z\in\mathbb{C}\quad\text{and}\quad|z|<1\}.$$

Then, following the pioneering work of MacGregor [18], we say that the function $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$ and write

$$f(z)\ll g(z)\qquad(z\in\mathbb{U}) \tag{4.1}$$

if there exists a function $\varphi(z)$, analytic in $\mathbb{U}$, such that

$$|\varphi(z)| \leqq 1 \quad \text{and} \quad f(z) = \varphi(z)\, g(z) \qquad (z \in \mathbb{U}). \tag{4.2}$$

The majorization (4.1) is closely related to the concept of *quasi-subordination* between analytic functions in $\mathbb{U}$, which was considered recently by (for example) Altintaş and Owa [5]. Altintaş *et al.* [8], on the other hand, investigated several majorization problems involving a number of subclasses of analytic functions in $\mathbb{U}$. In a sequel to the work of Altintaş *et al.* [8], we investigate the corresponding majorization problems associated with the classes $\mathcal{S}_{p,q}(\gamma)$ and $\mathcal{C}_{p,q}(\gamma)$ of p-valently starlike and p-valently convex functions of *complex* order $\gamma \neq 0$ in $\mathbb{U}$, which are introduced below (see, for details, Altintaş and Srivastava [10]).

Let $\mathcal{A}_p$ denote the class of functions f *normalized* by [*cf.* Definitions (1.1) *and* (3.1)]

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n\, z^n \qquad (p \in \mathbb{N} := \{1,2,3,\ldots\}), \tag{4.3}$$

which are analytic *and* p-valent in $\mathbb{U}$. Also let

$$\mathcal{A} := \mathcal{A}_1. \tag{4.4}$$

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_{p,q}(\gamma)$ of *p-valently starlike functions of complex order* $\gamma \neq 0$ in $\mathbb{U}$ if and only if

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{z f^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right)\right) > 0 \tag{4.5}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ \gamma \in \mathbb{C} \setminus \{0\};\ |2\gamma - p + q| \leqq p - q),$$

where, as usual, $f^{(q)}(z)$ denotes the derivative of $f(z)$ with respect to z of order $q \in \mathbb{N}_0$. Furthermore, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{C}_{p,q}(\gamma)$ of *p-valently convex functions of complex order* $\gamma \neq 0$ in $\mathbb{U}$ if and only if

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{z f^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q\right)\right) > 0 \tag{4.6}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ \gamma \in \mathbb{C} \setminus \{0\};\ |2\gamma - p + q| \leqq p - q).$$

Clearly, we have the following relationships:

$$\mathcal{S}_{1,0}(\gamma) = \mathcal{S}(\gamma) \qquad \text{and} \qquad \mathcal{C}_{1,0}(\gamma) = \mathcal{C}(\gamma) \qquad (\gamma \in \mathbb{C} \setminus \{0\}), \tag{4.7}$$

where $\mathcal{S}(\gamma)$ and $\mathcal{C}(\gamma)$ are the aforementioned classes of starlike and convex functions of *complex* order $\gamma \neq 0$ in $\mathbb{U}$, which were considered earlier by Nasr and Aouf [21] and Wiatrowski [30], respectively, and (more recently) by Altintaş *et al.* [8] (see also Aouf *et al.* [11]). Moreover, it is easily seen that

$$\mathcal{S}_{1,0}(1-\alpha) = \mathcal{S}(1-\alpha) = \mathcal{S}^*(\alpha) \qquad (0 \leqq \alpha < 1) \tag{4.8}$$

and

$$\mathcal{C}_{1,0}(1-\alpha) = \mathcal{C}(1-\alpha) = \mathcal{K}(\alpha) \qquad (0 \leqq \alpha < 1), \tag{4.9}$$

where $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote, respectively, the familiar classes of (normalized) starlike and convex functions of order α in $\mathbb{U}$, which were introduced by Robertson [23] (see also Srivastava and Owa [29]).

We first consider the majorization problems for the class $\mathcal{S}_{p,q}(\gamma)$, given by

Theorem 8. *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g \in \mathcal{S}_{p,q}(\gamma)$. If $f^{(q)}(z)$ is majorized by $g^{(q)}(z)$ in $\mathbb{U}$ for $q \in \mathbb{N}_0$, then*

$$\left|f^{(q+1)}(z)\right| \leqq \left|g^{(q+1)}(z)\right| \qquad (|z| \leqq r_1), \tag{4.10}$$

where

$$r_1 = r_1(p,q;\gamma) := \frac{\kappa - \sqrt{\kappa^2 - 4(p-q)|2\gamma - p + q|}}{2|2\gamma - p + q|} \tag{4.11}$$

$$(\kappa := 2 + p - q + |2\gamma - p + q|;\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ \gamma \in \mathbb{C} \setminus \{0\}).$$

Proof. Since $g \in \mathcal{S}_{p,q}(\gamma)$, we find from (4.5) that, if

$$h(z) := 1 + \frac{1}{\gamma}\left(\frac{zg^{(q+1)}(z)}{g^{(q)}(z)} - p + q\right) \qquad (\gamma \in \mathbb{C} \setminus \{0\}), \tag{4.12}$$

then

$$\Re\{h(z)\} > 0 \qquad (z \in \mathbb{U}) \tag{4.13}$$

and

$$h(z) = \frac{1 + w(z)}{1 - w(z)} \qquad (w \in \Omega), \tag{4.14}$$

where Ω denotes the well-known class of *bounded* analytic functions in $\mathbb{U}$, which satisfy the conditions (*cf.*, *e.g.*, Goodman [14, p. 58]):

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| \leqq |z| \qquad (z \in \mathbb{U}). \tag{4.15}$$

Making use of (4.12) and (4.14), we readily obtain

$$\frac{zg^{(q+1)}(z)}{g^{(q)}(z)} = \frac{p - q + (2\gamma - p + q)w(z)}{1 - w(z)}, \tag{4.16}$$

which, in view of (4.15), immediately yields the following inequality:

$$\left|g^{(q)}(z)\right| \leqq \frac{(1 + |z|)|z|}{p - q - |2\gamma - p + q| \cdot |z|}\left|g^{(q+1)}(z)\right| \qquad (z \in \mathbb{U}). \tag{4.17}$$

Next, since $f^{(q)}(z)$ is majorized by $g^{(q)}(z)$ in $\mathbb{U}$, from (4.2) we have

$$f^{(q+1)}(z) = \varphi(z)g^{(q+1)}(z) + \varphi'(z)g^{(q)}(z) \qquad (z \in \mathbb{U}). \tag{4.18}$$

Thus, observing that $\varphi \in \Omega$ satisfies the inequality (*cf.* Nehari [22, p. 168]):

$$|\varphi'(z)| \leqq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \qquad (z \in \mathbb{U}), \tag{4.19}$$

and applying (4.17) and (4.19) in (4.18), we get

$$\left|f^{(q+1)}(z)\right| \leqq \left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \cdot \frac{(1+|z|)\,|z|}{p-q-|2\gamma-p+q|\cdot|z|}\right) \cdot \left|g^{(q+1)}(z)\right| \qquad (z \in \mathbb{U}), \tag{4.20}$$

which, upon setting

$$|z| = r \qquad \text{and} \qquad |\varphi(z)| = \rho \qquad (0 \leqq \rho \leqq 1), \tag{4.21}$$

leads us to the following inequality:

$$\left|f^{(q+1)}(z)\right| \leqq \frac{\Theta(\rho)}{(1-r)\,(p-q-|2\gamma-p+q|\,r)} \left|g^{(q+1)}(z)\right| \qquad (z \in \mathbb{U}), \tag{4.22}$$

where the function $\Theta(\rho)$ defined by

$$\Theta(\rho) := -r\rho^2 + (1-r)\,(p-q-|2\gamma-p+q|\,r)\,\rho + r \qquad (0 \leqq \rho \leqq 1) \tag{4.23}$$

takes on its *maximum* value at $\rho = 1$ with

$$r = r_1(p,q;\gamma)$$

given by (4.11). Furthermore, if

$$0 \leqq \sigma \leqq r_1(p,q;\gamma),$$

where $r_1(p,q;\gamma)$ is given by (4.11), then the function $\Lambda(\rho)$ defined by

$$\Lambda(\rho) := -\sigma\rho^2 + (1-\sigma)\,(p-q-|2\gamma-p+q|\,\sigma)\,\rho + \sigma \tag{4.24}$$

is seen to be an *increasing* function on the interval $0 \leqq \rho \leqq 1$, so that

$$\Lambda(\rho) \leqq \Lambda(1) = (1-\sigma)\,(p-q-|2\gamma-p+q|\,\sigma)$$
$$(0 \leqq \rho \leqq 1;\ 0 \leqq \sigma \leqq r_1(p,q;\gamma)).$$

Hence, by setting $\rho = 1$ in (4.22), we conclude that the assertion (4.10) of Theorem 8 holds true for $|z| \leqq r_1(p,q;\gamma)$, where $r_1(p,q;\gamma)$ is given by (4.11). This evidently completes the proof of Theorem 8.

In view of the first relationship in (4.7), a special case of Theorem 8 when $p = 1$ and $q = 0$ yields

Corollary 3 (Altintaş *et al.* [8, p. 211, Theorem 1]). *Let the function $f(z)$ be in the class $\mathcal{A}$ and suppose that $g \in \mathcal{S}(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leqq |g'(z)| \qquad (|z| \leqq R_1), \tag{4.25}$$

where

$$R_1 = R_1(\gamma) := \frac{3 + |2\gamma-1| - \sqrt{9 + 2\,|2\gamma-1| + |2\gamma-1|^2}}{2\,|2\gamma-1|}. \tag{4.26}$$

Several further consequences of Corollary 3, involving such familiar classes as (see, for details, Duren [12] and Goodman [14])

$$\mathcal{S}^* := \mathcal{S}^*(0) \quad \text{and} \quad \mathcal{K} := \mathcal{K}(0) \tag{4.27}$$

were given earlier by MacGregor [18, p. 96, Theorems 1B and 1C] (see also Altintaş *et al.* [8, p. 213, Corollaries 1 and 2]).

The proof of our next result (Theorem 9 below), dealing with the majorization problems for the class $\mathcal{C}_{p,q}(\gamma)$, is based essentially upon the following result.

Lemma 5 (*cf.* Altintaş and Srivastava [10, p. 180, Lemma]). *If* $f \in \mathcal{C}_{p,q}(\gamma)$ $(\gamma \in \mathbb{C} \setminus \{0\})$, *then* $f \in \mathcal{S}_{p,q}\left(\frac{1}{2}\gamma\right)$, *that is,*

$$\mathcal{C}_{p,q}(\gamma) \subset \mathcal{S}_{p,q}\left(\frac{1}{2}\gamma\right) \qquad (\gamma \in \mathbb{C} \setminus \{0\}). \tag{4.28}$$

Proof. Since (*cf.*, *e.g.*, MacGregor [19, p. 71])

$$f \in \mathcal{K} \Longrightarrow f \in \mathcal{S}^*\left(\frac{1}{2}\right), \tag{4.29}$$

or, equivalently, since

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \Longrightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2} \qquad (z \in \mathbb{U}), \tag{4.30}$$

for $f(z) \longmapsto f^{(q)}(z)$ $(q \in \mathbb{N}_0)$ *with* $f \in \mathcal{A}_p$, we have

$$\begin{aligned} &\Re\left(1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - (p-q-1)\right) > 0 \\ &\Longrightarrow \Re\left(1 + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p-q)\right) > \frac{1}{2} \qquad (z \in \mathbb{U}), \end{aligned} \tag{4.31}$$

which readily yields the following assertion:

$$\begin{aligned} &1+\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q + 1 = \frac{1 - w(z)}{1 + w(z)} \\ &\Longrightarrow 1 + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q = \frac{1}{1 + w(z)} \qquad (w \in \Omega). \end{aligned} \tag{4.32}$$

Now, by making use of (4.32) appropriately, it is easily seen that

$$\begin{aligned} &1 + \frac{1}{\gamma}\left(1 + \frac{zf^{(q+2)}(z)}{f^{(q)}(z)} - p + q\right) = \frac{\gamma + (\gamma - 2)\,w(z)}{\gamma\,[1 + w(z)]} \\ &\Longrightarrow 1 + \frac{2}{\gamma}\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right) = \frac{\gamma + (\gamma - 2)\,w(z)}{\gamma\,[1 + w(z)]} \qquad (w \in \Omega), \end{aligned} \tag{4.33}$$

and the desired inclusion property (4.28) follows immediately from (4.33) in view of the characterizations (4.5) and (4.6) for the function classes $\mathcal{S}_{p,q}(\gamma)$ and $\mathcal{C}_{p,q}(\gamma)$, respectively.

Theorem 9. *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g \in \mathcal{C}_{p,q}(\gamma)$. If $f^{(q)}(z)$ is majorized by $g^{(q)}(z)$ in $\mathbb{U}$ for $q \in \mathbb{N}_0$, then*

$$\left|f^{(q+1)}(z)\right| \leqq \left|g^{(q+1)}(z)\right| \qquad (|z| \leqq r_2), \tag{4.34}$$

where

$$r_2 = r_2(p,q;\gamma) := \frac{\mu - \sqrt{\mu^2 - 4(p-q)|\gamma - p + q|}}{2|\gamma - p + q|} \tag{4.35}$$

$$(\mu := 2 + p - q + |\gamma - p + q|;\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ \gamma \in \mathbb{C} \setminus \{0\}).$$

Proof. In view of the inclusion property (4.28) asserted by Lemma 5, Theorem 9 can be deduced as a simple consequence of Theorem 8 with $\gamma \longmapsto \frac{1}{2}\gamma$.

By setting $p = 1$ and $q = 0$, Theorem 9 yields

Corollary 4 (Altintaş *et al.* [8, p. 214, Theorem 2]). *Let the function $f(z)$ be in the class $\mathcal{A}$ and suppose that $g \in \mathcal{C}(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leqq |g'(z)| \qquad (|z| \leqq R_2), \tag{4.36}$$

where

$$R_2 = R_2(\gamma) := \frac{3 + |\gamma - 1| - \sqrt{9 + 2|\gamma - 1| + |\gamma - 1|^2}}{2|\gamma - 1|}. \tag{4.37}$$

Finally, in its limit case when $\gamma \longrightarrow 1$, if we make use of the relationship [*cf.* Equations (4.9) *and* (4.27)]:

$$\mathcal{C}(1) = \mathcal{K}(0) =: \mathcal{K}, \tag{4.38}$$

Corollary 4 *further* yields

Corollary 5 (*cf.* MacGregor [18, p. 96, Theorem 1C]). *Let the function $f(z)$ be in the class $\mathcal{A}$ and suppose that $g \in \mathcal{K}$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leqq |g'(z)| \qquad \left(|z| \leqq \frac{1}{3}\right). \tag{4.39}$$

In view of the well-known inclusion property (4.29), Corollary 5 can also be deduced from Corollary 3 by letting $\gamma \longrightarrow \frac{1}{2}$ (see also Altintaş *et al.* [8, p. 213, Corollary 2]).

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